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λ-ANALOGUE OF STIRLING NUMBERS OF THE FIRST KIND

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ABSTRACT. In this paper, we consider the λ-analogue of the Stirling numbers of the first kind. In addition, we give some new identities and properties for these numbers.

1. Introduction

The falling factorials are given by

$$(x)_0 = 1,\ (x)_n = x(x-1)\cdots(x-n+1),\ (n \geq 1), \quad \text{(see [2,3,4])}. \tag{1.1}$$

The Stirling numbers of the first kind are defined as

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l,\ (n \geq 0), \quad \text{(see [5,9,10])}, \tag{1.2}$$

where $S_1(n,l)$, $(n,l \geq 0)$, are called the Stirling numbers of the first kind. We recall that the rising factorials are given by

$$< x >_0 = 1,\ < x >_n = x(x+1)(x+2)\cdots(x+n-1),\ (n \geq 1). \tag{1.3}$$

The unsigned Stirling numbers of the first kind are defined by

$$< x >_n = \sum_{l=0}^{n} |S_1(n,l)|x^l,\ (n \geq 0), \quad \text{(see [3])}. \tag{1.4}$$

From (1.3) and (1.4), we note that

$$(x)_n = \sum_{l=0}^{n} (-1)^{n-l}|S_1(n,l)|x^l, \quad \text{(see [3,5,10])}. \tag{1.5}$$

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It is well known that the generating function of the Stirling number of the first kind is given by

$$\frac{1}{k!}\big(\log(1+t)\big)^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \quad \text{(see [10])}. \tag{1.6}$$

The symbol $S_2(n,k)$ stands for the number of ways to partition a set of n things into k nonempty subsets. For example, there are seven ways to split a four-element set into two parts:

$$\{1,2,3\}\cup\{4\},\ \{1,2,4\}\cup\{3\},\ \{1,3,4\}\cup\{2\},\ \{2,3,4\}\cup\{1\},$$
$$\{1,2\}\cup\{3,4\},\ \{1,3\}\cup\{2,4\},\ \{1,4\}\cup\{2,3\};$$

thus $S_2(4,2)=7$.

The sequence $S_2(n,k)$, $(n,k\geq 0)$, are called the Stirling numbers of the second kind which are defined by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l,\ (n\geq 0), \quad \text{(see [1 - 10])}. \tag{1.7}$$

The generating function of $S_2(n,k)$ is defined by

$$\frac{1}{k!}\big(e^t-1\big)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, \quad \text{(see [10])}. \tag{1.8}$$

From (1.7) and (1.8), we have

$$S_2(n+1,k) = S_2(n,k-1) + kS_2(n,k),\ (1\leq k\leq n). \tag{1.9}$$

Indeed, $S_2(n,2) = 2^{n-1}-1$, $(n\geq 1)$. Note that $|S_1(n,k)|$, $(n,k\geq 0)$, counts the number of ways to arrange n objects into k cycles instead of subsets. From (1.5), we have

$$|S_1(n+1,k)| = |S_1(n,k-1)| + n|S_1(n,k)| \tag{1.10}$$

and

$$S_1(n+1,k) = S_1(n,k-1) + nS_1(n,k),\ (1\leq k\leq n). \tag{1.11}$$

In this paper, we consider the λ-analogue of the Stirling numbers of the first kind due to Carlitz and we give some new properties and identities for these numbers.

2. λ-analogue of Stirling numbers of the first kind

For $\lambda \in \mathbb{R}$, let us define the λ-analogue of falling factorials as follows:

$$(x)_{0,\lambda} = 1,\ (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda),\ (n \geq 1). \tag{2.1}$$

Note that $\lim_{\lambda\to 1}(x)_{n,\lambda} = (x)_n$, $(n \geq 0)$. and $\lim_{\lambda\to 0}(x)_{n,\lambda} = x^n$. From (2.1), we consider the λ-analogue of binomial coefficients which are given by

$$\binom{x}{n}_{\lambda} = \frac{(x)_{n,\lambda}}{n!} = \frac{x(x-\lambda)\cdots(x-(n-1)\lambda)}{n!},\ (n \geq 1),\ \binom{x}{0}_{\lambda} = 1. \tag{2.2}$$

For $n \in \mathbb{N}$, we define λ-analogue of $n!$ as follows:

$$(n!)_{\lambda} = n(n-\lambda)(n-2\lambda)\cdots(n-(n-1)\lambda) = (n)_{n,\lambda}. \tag{2.3}$$

Thus, by (2.2) and (2.3), we get

$$\binom{n}{k}_{\lambda} = \frac{(n!)_{\lambda}}{k!(n-k\lambda)_{n-k,\lambda}} = \frac{(n)_{k,\lambda}}{k!},\ (n \geq k \geq 0). \tag{2.4}$$

It is not difficult to show that the generating function of $\binom{x}{n}_{\lambda}$ is given by

$$(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}\binom{x}{n}_{\lambda} t^n. \tag{2.5}$$

By (2.5), we easily get

$$\sum_{m=0}^{n}\binom{y}{m}_{\lambda}\binom{x}{n-m}_{\lambda} = \binom{x+y}{n}_{\lambda},\ (n \geq 0). \tag{2.6}$$

Now, we define the λ-analogue of the Stirling numbers of the first kind as follows:

$$(x)_{n,\lambda} = \sum_{l=0}^{n} S_{1,\lambda}(n,l)x^l,\ (n \geq 0). \tag{2.7}$$

The coefficients $S_{1,\lambda}(n,l)$ on the right hand sides of (2.7) are called the λ-analogue of the Stirling numbers of the first kind. From (2.7), we note that

$$S_{1,\lambda}(0,0) = 1,\ S_{1,\lambda}(n,0) = S_{1,\lambda}(0,n) = 0,\ (n \in \mathbb{N}), \tag{2.8}$$

and

$$\lim_{\lambda\to 1} S_{1,\lambda}(n,l) = S_1(n,l) \quad \text{and} \quad \lim_{\lambda\to 0} S_{1,\lambda}(n,l) = \delta_{n,l},$$

where $\delta_{n,l}$ is the kronecker's symbol. From (2.7), we note that

$$\begin{aligned}\sum_{l=0}^{n+1} S_{1,\lambda}(n+1,l)x^l &= (x)_{n+1,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-n\lambda)\\ &= x(x-\lambda)\cdots(x-(n-1)\lambda)(x-n\lambda) = (x)_{n,\lambda}(x-n\lambda)\\ &= \sum_{l=0}^{n} S_{1,\lambda}(n,l)x^l(x-n\lambda)\\ &= \sum_{l=1}^{n+1} S_{1,\lambda}(n,l-1)x^l - n\lambda\sum_{l=1}^{n+1} S_{1,\lambda}(n,l)x^l,\ (n\in\mathbb{N}).\end{aligned} \tag{2.9}$$

Thus, (2.9), we get

$$S_{1,\lambda}(n+1,l) = S_{1,\lambda}(n,l-1) - n\lambda S_{1,\lambda}(n,l), \tag{2.10}$$

where $1\leq l\leq n$. Note that the generating function of $S_{1,\lambda}(n,k)$ is given by

$$\frac{1}{k!}\big(\log(1+\lambda t)^{\frac{1}{\lambda}}\big)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!}. \tag{2.11}$$

Now, we define the λ-analogue of unsigned Stirling numbers of the first kind as follows:

$$\begin{aligned}< x >_{n,\lambda} &= x(x+\lambda)(x+2\lambda)\cdots(x+(n-1)\lambda)\\ &= \sum_{l=0}^{n} |S_{1,\lambda}(n,l)|x^l,\ (n\geq 0).\end{aligned} \tag{2.12}$$

From (2.12), we note that

$$\begin{aligned}\sum_{l=0}^{n+1} |S_{1,\lambda}(n+1,l)|x^l &=< x >_{n+1,\lambda}= x(x+1)\cdots(x+n\lambda)\\ &= x(x+\lambda)\cdots(x+(n-1)\lambda)(x+n\lambda)\\ &= \sum_{l=0}^{n} |S_{1,\lambda}(n,l)|x^{l+1} + n\lambda\sum_{l=0}^{n}|S_{1,\lambda}(n,l)|x^l\\ &= \sum_{l=1}^{n+1} |S_{1,\lambda}(n,l-1)|x^l + n\lambda\sum_{l=1}^{n+1}|S_{1,\lambda}(n,l)|x^l.\end{aligned} \tag{2.13}$$

Thus, by (2.13), we get

$$|S_{1,\lambda}(n+1,l)| = |S_{1,\lambda}(n,l-1)| + n\lambda|S_{1,\lambda}(n,l)|, \tag{2.14}$$

where $1 \leq l \leq n$. We observe that

0	$S_{1,\lambda}(n,0)$	$S_{1,\lambda}(n,1)$	$S_{1,\lambda}(n,2)$	$S_{1,\lambda}(n,3)$	$S_{1,\lambda}(n,4)$	$S_{1,\lambda}(n,5)$	$S_{1,\lambda}(n,6)$	$\cdots$
0	1	0	0	0	0	0	0	$\cdots$
1	0	1	0	0	0	0	0	$\cdots$
2	0	$-\lambda$	1	0	0	0	0	$\cdots$
3	0	$2!\lambda^2$	-3λ	1	0	0	0	$\cdots$
4	0	$-3!\lambda^3$	$11\lambda^2$	-6λ	1	0	0	$\cdots$
5	0	$4!\lambda^4$	$-50\lambda^3$	$35\lambda^2$	-10λ	1	0	$\cdots$
6	0	$-5!\lambda^5$	$274\lambda^4$	$-225\lambda^3$	$85\lambda^2$	-15λ	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Note that $S_{1,\lambda}(n,1) = (-1)^{n-1}\lambda^{n-1}(n-1)!$

0	$\lvert S_{1,\lambda}(n,0)\rvert$	$\lvert S_{1,\lambda}(n,1)\rvert$	$\lvert S_{1,\lambda}(n,2)\rvert$	$\lvert S_{1,\lambda}(n,3)\rvert$	$\lvert S_{1,\lambda}(n,4)\rvert$	$\lvert S_{1,\lambda}(n,5)\rvert$	$\lvert S_{1,\lambda}(n,6)\rvert$	$\cdots$
0	1	0	0	0	0	0	0	$\cdots$
1	0	1	0	0	0	0	0	$\cdots$
2	0	λ	1	0	0	0	0	$\cdots$
3	0	$2!\lambda^2$	3λ	1	0	0	0	$\cdots$
4	0	$3!\lambda^3$	$11\lambda^2$	6λ	1	0	0	$\cdots$
5	0	$4!\lambda^4$	$50\lambda^3$	$35\lambda^2$	10λ	1	0	$\cdots$
6	0	$5!\lambda^5$	$274\lambda^4$	$225\lambda^3$	$85\lambda^2$	15λ	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Note that $|S_{1,\lambda}(n,1)| = \lambda^{n-1}(n-1)!$.

From (2.7) and (2.12), we have

$$\begin{aligned}\sum_{l=0}^{n}|S_{1,\lambda}(n,l)|x^l &=< x >_{n,\lambda}= (-1)^n(-x)_{n,\lambda}\\ &= (-1)^n\sum_{l=0}^{n}S_{1,\lambda}(n,l)(-x)^l \\ &= \sum_{l=0}^{n}S_{1,\lambda}(n,l)(-1)^{n-l}x^l.\end{aligned} \tag{2.15}$$

Thus, by (2.15), we get

$$|S_{1,\lambda}(n,l)| = (-1)^{n-l}S_{1,\lambda}(n,l),\ (n \geq 0). \tag{2.16}$$

Now, we observe that

$$\begin{aligned}(x)_{n,\lambda} &= \sum_{l=0}^{n} S_{1,\lambda}(n,l)x^l = \sum_{l=0}^{n} S_{1,\lambda}(n,l)\lambda^l \left(\tfrac{x}{\lambda}\right)^l \\ &= \sum_{l=0}^{n} S_{1,\lambda}(n,l)\lambda^l \sum_{m=0}^{l} S_2(l,m) \left(\tfrac{x}{\lambda}\right)_m \\ &= \sum_{l=0}^{n} S_{1,\lambda}(n,l)\lambda^l \sum_{m=0}^{l} S_2(l,m)\lambda^{-m}(x)_{m,\lambda} \\ &= \sum_{m=0}^{n} \left(\sum_{l=m}^{n} S_{1,\lambda}(n,l)\lambda^{l-m} S_2(l,m) \right)(x)_{m,\lambda}.\end{aligned} \tag{2.17}$$

Thus, by (2.17), we get

$$\sum_{l=0}^{n} S_{1,\lambda}(n,l)S_2(l,m)\lambda^{l-m} = \delta_{n,m}, \tag{2.18}$$

where $0 \le m \le l \le n$. On the other hand, we have

$$\begin{aligned}\left(\frac{x}{\lambda}\right)^n &= \sum_{l=0}^{n} S_2(n,l) \left(\tfrac{x}{\lambda}\right)_l = \sum_{l=0}^{n} S_2(n,l)\lambda^{-l}x(x-\lambda)\cdots(x-(n-1)\lambda) \\ &= \sum_{l=0}^{n} S_2(n,l)\lambda^{-l}(x)_{l,\lambda} = \sum_{l=0}^{n} S_2(n,l)\lambda^{-l} \sum_{m=0}^{l} S_{1,\lambda}(l,m)x^m \\ &= \sum_{m=0}^{n} \left(\sum_{l=m}^{n} S_2(n,l)S_{1,\lambda}(l,m)\lambda^{-l} \right) x^m.\end{aligned} \tag{2.19}$$

Thus, by (2.19), we get

$$x^n = \sum_{m=0}^{n} \left(\sum_{l=m}^{n} S_2(n,l)S_{1,\lambda}(l,m)\lambda^{n-l} \right) x^m. \tag{2.20}$$

Therefore, from (2.20), we have

$$\sum_{l=m}^{n} S_2(n,l)S_{1,\lambda}(l,m)\lambda^{n-l} = \delta_{m,n}, \tag{2.21}$$

where $0 \le m \le l \le n$.

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